

B. Sc. Examination by Course Unit 2008

MAS 317 Linear Algebra II

Duration: 2 hours

Date and time: SAMPLE PAPER

You may attempt as many questions as you wish and all questions carry equal marks, Except for the award of a bare pass, only the best FOUR questions answered will be counted. YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE INVIGILATOR.

Question 1 (a) [4 marks] Let $u_1, \dots, u_k$ be elements of a vector space V over a field F . Give the definition of $\langle u_1, \dots, u_k \rangle$, the subspace spanned by $u_1, \dots, u_k$.

What is meant by saying that $u_1, \dots, u_k$ are linearly independent?

(b) [6 marks] Let $v_1, \dots, v_n$ be elements of a vector space V over a field F .

Prove that, if $0 \leq r < n$ and $v_1, \dots, v_r$ are linearly independent and $v_1, \dots, v_n$ are linearly dependent, then

$\langle v_1, \dots, v_n \rangle = \langle v_1, \dots, v_{t-1}, v_{t+1}, \dots, v_n \rangle$ for some t with $r < t \leq n$.

(c) [4 marks] State the Steinitz Exchange Lemma. and use it to prove that any two bases for a finite-dimensional vector space contain the same number of vectors.

(d) [8 marks] Find a basis for the row space of the matrix A , where

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & -1 & -2 \\ 2 & 1 & 8 & -6 & -2 & -7 \\ 3 & 2 & 11 & -3 & 2 & -15 \end{pmatrix}$$

(e) [3 marks] What is the column rank of this matrix A ? State any result that you use to obtain your answer.

Question 2 V is a vector space of dimension n over a field F , and $\mathcal{B}$ is an ordered basis for V , and f is a bilinear form on V .

- (a) [2 marks] Explain what is meant by saying that f is (i) *symmetric* and (ii) *alternating*.
- (b) [7 marks] Give the definition of the matrix $(f; \mathcal{B})$ of f with respect to $\mathcal{B}$.

Prove that

- (i) if f is symmetric then $(f; \mathcal{B})$ is a symmetric matrix, and
(ii) if f is alternating then $(f; \mathcal{B})$ is an alternating matrix

- (c) [9 marks] Let U be a subspace of V . Give the definition of $U^\perp$. What is meant by saying that U is *non-degenerate*?

Let $u_1, \dots, u_k$ be a basis for U , and let A be the $k \times k$ matrix with ij -entry equal to $f(u_i, u_j)$ for all i, j . Prove that if U is not non-degenerate then A has row rank less than k .

- (d) [7 marks] The bilinear form $f : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is defined by

$$f((x_1, x_2, x_3), (y_1, y_2, y_3)) = x_1y_1 + 3x_1y_2 + 2x_2y_1 - x_2y_3 + 6x_3y_2 + x_3y_3.$$

Write down the matrix $(f; \mathcal{E})$, where $\mathcal{E}$ is the standard basis for $\mathbb{R}^3$, and use this to decide whether $\mathbb{R}^3$ is non-degenerate w.r.t. f . Explain your answer.

Let $U = \langle (1, 0, 1), (2, 1, 0) \rangle$. Is U non-degenerate w.r.t. f ? Explain your answer.

Question 3 (a) [9 marks] Let A, B, C be $n \times n$ matrices over a field F . What is meant by saying that A is *congruent* to B ?

Prove that if A and C are both congruent to B then A is congruent to C .

- (b) [7 marks] State the *Normal Form Theorem* for symmetric matrices over $\mathbb{C}$.

Are the following two matrices congruent as matrices over $\mathbb{C}$? Give a proof of your answer.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{pmatrix}.$$

(c) [9 marks] Let $A = \begin{pmatrix} 4 & 4 & -2 \\ 4 & 3 & 2 \\ -2 & 2 & 1 \end{pmatrix}$.

Find the normal form D for A over $\mathbb{R}$ and a matrix P such that $P^T A P = D$ by answering one of (i) and (ii) as follows (and NOT by using row and column operations):

(i) Let f be the bilinear form on $\mathbb{R}^3$ such that the matrix of f with respect to the standard basis for $\mathbb{R}^3$ is A . Find a basis for $\mathbb{R}^3$ which is orthogonal with respect to f and hence find D and P .

OR

(ii) Write down the quadratic form over $\mathbb{R}$ corresponding to A and use the method of 'completing the square' to find D and P^{-1} . This is sufficient, you need not find P .

Question 4 (a) [12 marks] State the *normal form theorem for alternating matrices under congruence*.

Describe in detail, without proofs, a method for finding a matrix P such that $P^T A P$ is in normal form when A is an alternating matrix.

(b) [6 marks] Prove that alternating matrices have even row rank.

Are the following matrices congruent over $\mathbb{R}$? Explain your answer.

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & -5 \\ -2 & -1 & 0 & -11 \\ -1 & 5 & 11 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & 3 & 0 \\ -2 & 0 & 6 & -2 \\ -3 & -6 & 0 & -3 \\ 0 & 2 & 3 & 0 \end{pmatrix}.$$

(c) [7 marks] State the *normal form theorem for Hermitian matrices under Hermitian congruence*.

Find the normal form under Hermitian congruence for the matrix

$$\begin{pmatrix} 2 & 1+3i \\ 1-3i & 2 \end{pmatrix}.$$

Question 5 Let V be a vector space over $\mathbb{R}$ with an inner product $\langle \cdot, \cdot \rangle$ on V .

(a) [8 marks] Give the following definitions.

- (i) $\langle \cdot, \cdot \rangle$ is an *inner product* on V ,
- (ii) $\|u\|$ for $u \in V$,
- (iii) A set of vectors $\{u_1, \dots, u_m\}$ in V is *orthonormal*.
- (iv) The *orthogonal projection* $\pi_W(u)$ of $u \in V$ onto a subspace W of V .

(b) [8 marks] Prove that if $u, v \in V$ and u is orthogonal to v then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2,$$

and that, if W is a subspace of V , then

$$\|u - \pi_W(u)\| \leq \|u - w\| \text{ for all } w \in W.$$

(c) [9 marks]

Let $V = \mathbb{R}^4$ with the dot product as inner product, let

$$W = \langle (1, -1, 1, -1), (1, 1, 0, 0), (0, 0, 1, 1) \rangle.$$

Prove that these three vectors are orthogonal.

Let $v = (4, 1, -6, 1) \in \mathbb{R}^4$. Find the point $w \in W$ having the minimum distance from v . What is this distance?

Question 6 Let V be a complex inner product space with $\alpha : V \rightarrow V$ a self-adjoint linear map. Let λ and μ be distinct (real) eigenvalues of α with eigenvectors u and v respectively.

- (a) [1 mark] Explain what is meant by the term *self-adjoint*.
- (b) [4 marks] Show that u and v are orthogonal.
- (c) [4 marks] Prove that if $w \in \langle u \rangle^\perp$ then $\alpha(w) \in \langle u \rangle^\perp$, for the eigenvector vector u .
- (d) [6 marks] Prove that there exists a basis $\mathcal{B}$ for V which is orthonormal with respect to $\langle \cdot, \cdot \rangle$ and consists of eigenvectors for α .
- (e) [10 marks] Find a unitary matrix U and a diagonal matrix D such that $U^T A U = U^{-1} A U = D$ when

$$A = \begin{pmatrix} 0 & i & -i \\ -i & 0 & 1 \\ i & 1 & 0 \end{pmatrix}.$$